

PS9

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1. Let R be an integral domain and let $P \subset R$ be a prime ideal.
 - (a) Prove that $D = R \setminus P$ is a multiplicative subset.
 - (b) Find all units in $Q = R[D^{-1}]$.
 - (c) Prove that Q has exactly one maximal ideal. (A ring is called **local** if it has exactly one maximal ideal.)

(a) Assume P is a proper ideal to avoid trivial case, then we have $1 \in D = R \setminus P$. Let $a, b \in D$, then $ab \in D$ because neither a nor b is in P so then $ab \notin P$, and since R is an integral domain, D has no zero divisors.

(b) $Q = \{\frac{r}{d} : r \in R, d \in D\}$. For $\frac{r_1}{d_1} \in Q$ to be a unit there must exist $\frac{r_2}{d_2} \in Q$ such that $\frac{r_1}{d_1} \cdot \frac{r_2}{d_2} = \frac{r_1 r_2}{d_1 d_2} = 1$ this is true if $r_1 r_2 = d_1 d_2$, since D is closed under multiplication, $r_1 r_2$ must be in D . For $r_1 r_2$ to be in $D = R \setminus P$ we must have $r_1, r_2 \in D$ because if one of them was in P then $r_1 r_2$ would be in P (P prime), so $Q^\times = \{\frac{d_1}{d_2} : d_1, d_2 \in D\}$.

(c) $M = \{\frac{s}{d} : s \in P, d \in D\}$ is a maximal ideal: Suppose there is an ideal J such that $M \subsetneq J \subseteq R$, then there exists $\frac{d_1}{d_2} \in J \setminus M$ but $d_1, d_2 \in D$ so $\frac{d_1}{d_2}$ is a unit and thus $\exists \frac{q_2}{q_1} \in Q$ such that

$$\frac{q_1}{q_2} \cdot \frac{q_2}{q_1} = \frac{q_1 q_2}{q_2 q_1} = 1 \in J \implies J = R$$

For uniqueness, suppose there exists another maximal ideal I , then we must have $I \cap M = \{0\}$ because otherwise there exists $\frac{s}{a} \in J$ such that $\frac{r}{d} \cdot \frac{s}{a} \in J \cap M$ for any $\frac{r}{d} \in Q$ but then $M \subseteq I$, so M not maximal, contradiction. So $I \cap M = \{0\}$, then for $\frac{d_1}{d_2} \in I$, d_1, d_2 must be in D , but this is a unit, so $1 \in I \implies I = R$. Thus Q has only one maximal ideal.

2. Let R be a finite ring.

- (a) Prove that if R is an integral domain, then R is a field.
- (b) Prove that if $I \subset R$ is a prime ideal, then I is maximal.

(a) Let R be an integral domain. Suppose R is not a field, then there exists an element $x \in R^\times$ such that x has no multiplicative inverse. Then we can take x^n for any $n \in \mathbb{N}$

and generate a new element (because x^n will never be 0 or 1), but this contradicts the finiteness of R , so R must be a field.

- (b) Suppose I not maximal, then there exists an ideal J such that $I \subsetneq J \subseteq R$. Let $a \in J \setminus I$, then we can take a^n for any n and have $a^n \in J \setminus I$ because J is an ideal and I is a prime ideal, but since $0 \in I$, $0 \notin J \setminus I$ so $a^n \neq 0 \forall n$ so since R is finite, a^n must equal 1 for some $n \in \mathbb{N}$ so $1 \in J \implies J = R$ and then I is maximal.
3. Let R be the ring of all continuous functions from $[0, 1]$ to $\mathbb{R}$. For $c \in [0, 1]$, let $M_c = \{f \in R : f(c) = 0\}$.
- (a) Prove that for any $c \in [0, 1]$, M_c is a maximal ideal.
- (b) Prove that if $c, d \in [0, 1]$, then $M_c \neq M_d$.
- (c) Prove that if $I \subset R$ is any proper ideal, then $I \subseteq M_c$ for some $c \in [0, 1]$. [Hint: suppose not, and then for each $c \in [0, 1]$, we get $f_c \in I$ such that $f_c(c) \neq 0$. Use these functions and compactness to construct $f \in I$ such that $f(c) \neq 0$ for all $c \in [0, 1]$.]
- (d) Prove that $c \mapsto M_c$ establishes a bijection between $[0, 1]$ and the set of maximal ideals of $[0, 1]$.

- (a) Consider the ring homomorphism $\varphi_c : R \rightarrow \mathbb{R}$ defined by $\varphi_c(f) = f(c)$. This is a homomorphism because for $f, g \in R$:

$$\varphi_c(f + g) = (f + g)(c) = f(c) + g(c) = \varphi_c(f) + \varphi_c(g)$$

$$\varphi_c(f \cdot g) = (f \cdot g)(c) = f(c) \cdot g(c) = \varphi_c(f)\varphi_c(g)$$

And we have $\varphi_c(1) = 1$ and $\varphi_c(0) = 0$. $\text{Ker}(\varphi_c) = M_c$ and $\text{Im}(\varphi_c) = \mathbb{R}$ so by first isomorphism theorem for rings $R/M_c \cong \mathbb{R}$ and $\mathbb{R}$ is a field so R/M_c is a field too so M_c is maximal.

- (b) Let $c, d \in [0, 1]$, $c \neq d$ then $(x - c) \in M_c \setminus M_d$ so $M_c \neq M_d$.
- (c) Suppose not, then for each $c \in [0, 1]$, $f_c \in I : f_c(c) \neq 0$. Since f_c is continuous there exists an interval I_c where $f_c(x) \neq 0$ for all $x \in I_c$. Since we can do this for every $c \in [0, 1]$, $f_c \in R$,

$$\bigcup_{c \in [0, 1]} I_c = [0, 1]$$

And since $[0, 1]$ compact, we can cover it with finitely many I_{c_i} , enumerate these with I_{c_i} such that

$$\bigcup_{i \in \{1, \dots, n\}} I_{c_i} = [0, 1]$$

Then let

$$f(x) = \sum_{i=1}^n [f_{c_i}(x)]^2$$

Since this is a finite sum of elements in I , $f(x) \in I$ as well, and for any $x \in [0, 1]$ there is a $f_{c_i}(x) \neq 0$ so $f(x) > 0 \forall x \in [0, 1]$ so there exists an inverse $\frac{1}{f(x)} \in R$ but then $f(x) \cdot \frac{1}{f(x)} = 1 \in I$ but then I is not a proper ideal, so we must have $I \subseteq M_c$.

(d) Injective: for $c, d \in [0, 1]$, $c \neq d$, from (b) $M_c \neq M_d$.

Surjective: From (a) and (c) we know the only maximal ideals are $M_c : c \in [0, 1]$, so for any maximal ideal I , $I = M_c$ for $c \in [0, 1]$.

4. Let R be a ring, Prove that the ideal $\langle 0 \rangle = \{0\}$ is prime if and only if R is an integral domain. Assume $\langle 0 \rangle = \{0\}$ is prime. Then for any $ab \in \langle 0 \rangle$, $ab = 0$, $a = 0$ or $b = 0$ so R is an integral domain.

Assume R is an integral domain so for $ab \in \langle 0 \rangle$, $ab = 0$ so $a = 0$ or $b = 0$ thus $\langle 0 \rangle$ is prime.

5. The **Krull dimension** d of a ring R is the length of the longest possible chain of prime ideals $P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_d \subsetneq R$ in R .

(a) Prove that if F is a field, then $F[x_1, \dots, x_n]$ has a Krull dimension at least n .

(b) Prove that $\mathbb{Z}[x]$ has Krull dimension at least 2.

(a) We have that $\langle 0 \rangle = \{0\}$ is a prime ideal because $F[x_1, \dots, x_n]$ is an integral domain (F is an integral domain). Let $\langle 0 \rangle = P_0$ and $P_i = \langle x_1, \dots, x_i \rangle$ for $i \in \{1, \dots, n\}$. Since $x_{k+1} \in P_{k+1} \setminus P_k$ for some $k \in \{0, \dots, n-1\}$, we have:

$$P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_n \subsetneq R$$

To show prime, let $ab \in P_k$ for some $k \in \{0, \dots, n\}$, then $ab = \sum_{i=1}^k a_i x_i$ where $a_i \in F[x_1, \dots, x_n]$. Suppose neither a nor b is in P_k , then they must be in a larger prime ideal in the chain, then W.L.O.G suppose $a \in P_m \setminus P_k$ for $k < m \leq n$, but then a would have terms with x_m , but then ab would also have terms with x_m , this is a contradiction so a or b must be in P_k so P_k is prime for arbitrary k . So $F[x_1, \dots, x_n]$ has Krull dimension at least n .

(b) For $P_0 = \langle 0 \rangle$, $P_1 = \langle x \rangle$, and $P_2 = \langle x, 5 \rangle$ we have:

$$P_0 \subsetneq P_1 \subsetneq P_2 \subsetneq R$$

We know $\langle x \rangle$ is prime and $\langle x, 5 \rangle$ is prime because for $ab \in \langle x, 5 \rangle$, ab is a polynomial with constant term divisible by 5, and since 5 is prime, either a or b must have a constant term that is divisible by 5. Thus the Krull dimension of $\mathbb{Z}[x]$ is at least 2.