

PS7

Sasha Redden

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1. Let G be a group. For any $g, h \in G$, let $[g, h] = ghg^{-1}h^{-1}$. Let G' be the subgroup of G generated by $\{[g, h] : g, h \in G\}$. It is called the **commutator subgroup**.

- (a) Prove that G' is a normal subgroup of G .
- (b) Find a group G such that $G' = G$.
- (c) Find D'_5 .
- (d) Prove G/G' is abelian. It is called the **abelianization** of G .
- (e) Prove that if A is any abelian group and $\phi : G \rightarrow A$ is any group homomorphism, then there exists $\phi' : G/G' \rightarrow A$ such that $\phi = \phi' \circ \pi$ where $\pi : G \rightarrow G/G'$ is the usual projection.

- (a) $u \in G', g \in G$ then $gug^{-1} = uu^{-1}gug^{-1} = u[u^{-1}, g] \in G'$ thus G' is normal.
- (b) Since G' is normal in G , and $G' = \{e\}$ iff G is abelian, so if G is simple and not abelian, $G = G'$. Examples are $G = A_n$ for $n \geq 5$.
- (c) The (non-trivial) normal subgroups of D_n are $\langle r^d \rangle : d|n$ but since $n = 5$ is prime, the normal subgroups are $\langle r \rangle$ or D_n itself, but since the commutator with never form a rotation $D'_5 = \langle r \rangle$.
- (d) $G/G' = \{gG' : g \in G\}$. Because G' is normal in G , there is a group homomorphism $\varphi : G \rightarrow G/G', \varphi(g) = gG'$, then for some $g, h \in G/G$ we have $\varphi(g)\varphi(h) = gG' \cdot hG' = ghG' = \varphi(gh)$. So for any $g, h \in G$

$$gG' \cdot hG' = ghG' = gh[h^{-1}, g^{-1}]G' = ghgh^{-1}g^{-1}hgG' = hgG'$$

- (e) Let $\phi' : G/G' \rightarrow A$ be defined as $\phi'(gG') = \phi(g)$. Since $\phi : G \rightarrow A$ is a group homomorphism, A is abelian, and G' is normal in $G, \forall gG', hG' \in G/G',$

$$\phi'(hG' \cdot gG') = \phi'(gG' \cdot hG') = \phi'(ghG') = \phi(gh) = \phi(g)\phi(h) = \phi(h)\phi(g) = \phi'(hG') \cdot \phi'(gG')$$

2. Let n be an odd natural number. Let $G = D_{2n}$ and let $H = D_n$ and $K = \mathbb{Z}/2$. Explain how to realize H and K as normal, complementary subgroups of G . Conclude that $D_{2n} \cong D_n \times \mathbb{Z}/2$.

Realize $r \in H$ as $r^2 \in G$ and $s \in H$ as $s \in G$. Since $r, s \in H$ generate H this is a complete realization of H , and $\langle r^2, s \rangle$ is normal in $D_{2n} = G$. Realize $1 \in \mathbb{Z}/2$ as $r^n \in G$, since $n|2n$, $\langle r^n \rangle$ is normal in G . Since n is odd, $\langle r^2, s \rangle \cap \langle r^n \rangle = \{e\}$ and with r^2, r^n we can generate r . This realization of H and K in G are normal, complementary subgroups so thus $D_{2n} \cong D_n \times \mathbb{Z}/2$

3. Let G be a group, let $H \trianglelefteq G$, and let $K = G/H$. In this case we say that G is an **extension** of K by H and we draw a little picture

$$1 \rightarrow H \rightarrow G \rightarrow K \rightarrow 1$$

which is called a **short exact sequence** of groups. (This is just notation, the only information is G and H .)

- (a) We say that the short exact sequence is **split** if there is an homomorphism $\phi : K \rightarrow G$ such that $\pi \circ \phi = id_K$. (Here $\pi : G \rightarrow G/H = K$ is the usual projection.) Prove that if the short exact sequence is split, then G is isomorphic to the semi-direct product $H \rtimes K$.
- (b) Find a non-split extension of $\mathbb{Z}/2$ by $\mathbb{Z}/2$.

- (a) W.T.S $K = G/H \leq G$.

$$e = \phi(H) = \phi(gg^{-1}H) = \phi(gH \cdot g^{-1}H) = \phi(gH)\phi(g^{-1}H) = gg^{-1} = e$$

Where the third equality is due to the normality of H and the fourth is because ϕ is a homomorphism, so we must have that $\forall gH \in K, g^{-1}H \in K$. We also have that $\forall g_1H, g_2H, g_1H \cdot g_2H = g_1g_2H \in K$ because H is normal, so $K \leq G$ and because H and K are complementary, we have that $G \cong H \rtimes K$.

- (b) Since we know $\mathbb{Z}/2 \times \mathbb{Z}/2$ is not isomorphic to $\mathbb{Z}/4$, we can deduce that the extension of $\mathbb{Z}/2$ by $\mathbb{Z}/2$

$$1 \rightarrow \mathbb{Z}/2 \rightarrow \mathbb{Z}/4 \rightarrow \mathbb{Z}/2 \rightarrow 1$$

where the homomorphism from $\mathbb{Z}/2$ to $\mathbb{Z}/4$ is given by a multiplication by 2 and the homomorphism from $\mathbb{Z}/4$ to $\mathbb{Z}/2$ by modulo 2, is not split because the (in this case) direct product $\mathbb{Z}/2 \times \mathbb{Z}/2$ is not isomorphic to $\mathbb{Z}/4$.

4. Let H be a group and let $\alpha : H \rightarrow H$ be an automorphism. Prove that there exists a group G containing H as a normal subgroup and an element $g \in G$ such that $ghg^{-1} = \alpha(h)$ for all $h \in H$. (In other words, prove that every automorphism can be realized by conjugation after passing to a bigger group.) For any semi-direct product with H : $G = K \rtimes H$, we have $H \triangleleft G$. Consider $K = \text{Aut}(H)$ and $\varphi : K \rightarrow \text{Aut}(H)$ such that $\varphi = id_K$. Define semi-direct product $G = H \rtimes \text{Aut}(H)$ (called the *holomorph* of H) with multiplication as follows: $(h_1, k_1)(h_2, k_2) = (h_1\varphi_{k_1}(h_2), k_1k_2) = (h_1(k_1 \cdot h_2), k_1k_2)$. Then for $g = (e, \alpha) \in G, g^{-1} = (e, \alpha^{-1})$ so for $h \in H, (h, id) \in G$ where id is the trivial automorphism, we have $ghg^{-1} = (e, \alpha)(h, id)(e, \alpha^{-1}) = (e, \alpha)(h\varphi_{id}(e), id \circ \alpha^{-1}) = (e, \alpha)(h, \alpha^{-1}) = (e\varphi_\alpha(h), \alpha\alpha^{-1}) = (\alpha(h), id)$.
5. Let $H = \mathbb{Z}^n$, regarded as the group under pointwise addition. Let $K = S_n$. Let K act on H by $\sigma \cdot (a_1, \dots, a_n) = (a_{\sigma(1)}, \dots, a_{\sigma(n)})$.

(a) Prove that K acts on H by group automorphisms.

(b) Let $G = H \rtimes K$. Define a faithful action of G on $\mathbb{Z}^n$.

In fact, for any group G , we can define an action of S_n on G^n in this way, and the resulting group is called the **wreath product** of G by S_n .

(a) Action induces a homomorphism from H to H : for $(a_1, \dots, a_n), (b_1, \dots, b_n) \in \mathbb{Z}^n, \sigma \in S_n$ we have $\sigma \cdot [(a_1, \dots, a_n) + (b_1, \dots, b_n)] = \sigma \cdot (a_1 + b_1, \dots, a_n + b_n) = (a_{\sigma(1)} + b_{\sigma(1)}, \dots, a_{\sigma(n)} + b_{\sigma(n)}) = \sigma \cdot (a_1, \dots, a_n) + \sigma \cdot (b_1, \dots, b_n)$

Action is injective: $(a_1, \dots, a_n), (b_1, \dots, b_n) \in \mathbb{Z}^n : (a_1, \dots, a_n) \neq (b_1, \dots, b_n) \implies \exists i : a_i \neq b_i, \sigma \in S_n$

$\exists i : a_i \neq b_i \implies \text{for } j : \sigma(j) = i \text{ then } a_{\sigma(j)} \neq b_{\sigma(j)} \implies \sigma \cdot (a_1, \dots, a_n) \neq \sigma \cdot (b_1, \dots, b_n)$

Action is surjective: for any $(a_1, \dots, a_n) \in \mathbb{Z}^n, \exists (a_{x_1}, \dots, a_{x_n}) \in \mathbb{Z}^n : \sigma(x_i) = i \forall i$ and thus $\sigma \cdot (a_{x_1}, \dots, a_{x_n}) = (a_1, \dots, a_n)$.

(b) For $((a_1, \dots, a_n), \sigma) \in G, (b_1, \dots, b_n) \in \mathbb{Z}^n$ let $((a_1, \dots, a_n), \sigma) \cdot (b_1, \dots, b_n) = \sigma \cdot (a_1, \dots, a_n) + (b_1, \dots, b_n)$. This action is faithful because the only element in G that would leave every element in $\mathbb{Z}^n$ unchanged is the identity $((0, \dots, 0), ())$.