

PS6

Question 1. Let G be a finite group and p be a prime. Let $|G| = p^k m$ where $k \in \mathbb{N}$ and $p \nmid m$. Let Y_p be the set of Sylow p -subgroups: G acts on Y_p by conjugation. Part 2 of Sylow's theorem tells us that G acts transitively on Y_p .

Let $n_p = |Y_p|$ be the number of p -Sylow subgroups. Part 3 of Sylow's (not proved in class) tells us that $n_p \equiv 1 \pmod{p}$.

a) Let $P \leq G$ be a p -Sylow subgroup. Regard P as an element of the set Y_p . Prove that the stabilizer of P for the action of G on Y_p is $N_G(P)$.

b) Prove that $n_p | m$.

a)

$$N_G(P) = \{g \in G : \forall h \in P, ghg^{-1} \in P \text{ and } g^{-1}hg \in P\} = \{g \in G : gPg^{-1} = P = g^{-1}Pg\}$$

G acts on Y_p : $\forall g \in G$ $g \circ P = gPg^{-1}$, then $\text{Stab}_G(P) = \{g \in G : gPg^{-1} = P\}$. Let $g \in \text{Stab}_G(P)$, then $gPg^{-1} = P \implies gP = Pg \implies P = g^{-1}Pg$ thus $g \in N_G(P)$. The other implication is trivial!

b) If we consider the conjugation action of G on Y_p , since the Sylow p -subgroup form an equivalency class under conjugation and thus form an orbit; n_p is equal to the size of the orbit of the Sylow p -subgroups. By orbit stabilizer, this must divide the order of the group, so $n_p | p^k m = |G|$, but $n_p \equiv 1 \pmod{p}$, then $n_p \nmid p$ so n_p will not divide any power of p , thus $n_p | m$, since m and p^k are co-prime.

Question 2. Let G be a group and let p, q be two distinct primes. Let P be a p -subgroup of G and Q be a q -subgroup of G . Prove that $P \cap Q = \{e\}$.

$P \cap Q$ as an intersection of subgroups is itself a subgroup of P and Q . Then the order of this subgroup must divide $|P| = p$ and $|Q| = q$ and since p, q are two distinct primes, their only common divisor is 1, thus the intersection must be the identity, $P \cap Q = \{e\}$.

Question 3. Prove that a group of order 132 has a normal Sylow p -subgroup for some p . (Hint: determine the possible values of n_2, n_3, n_{11} using Q1, and then use Q2 to see that G must end up with too many elements.)

$132 = 2^2 \circ 3 \circ 11$. We know that $n_p \equiv 1 \pmod{p}$ and that $n_p | m$. Possible values for n_2 are all odd numbers that also divide 33, $n_2 = 1, 3, 11, 33$. Possible values for n_3 are 1, 4, 7, .. that also divide 44, so $n_3 = 1, 4$. Finally, possible values for n_{11} are 1, 12, 23, 34, 44 that also divide 12, so $n_{11} = 1, 12$. For a Sylow p -subgroup to be normal, $n_p = 1$ because then the subgroup is invariant under conjugation, $gPg^{-1} = P$. Assume that the group of order 132 has no normal Sylow p -subgroups, then $n_p \neq 1$ for $p = 2, 3, 11$. Assume n_p is the second smallest option, $n_2 = 3, n_3 = 4, n_{11} = 12$, then

$$132 = |G| \geq \sum_p n_p(p^k - 1) + 1 = 3(4 - 1) + 4(3 - 1) + 12(11 - 1) + 1 = 9 + 8 + 120 + 1 = 138$$

This is a contradiction, and the sum would only be larger if we take any of the larger options for n_p , so a group of order 132 must have a normal Sylow p -subgroup.