

PS5

Question 1. Find all conjugacy classes in D_n and write out its class equation. Careful, your answer will look different depending on whether n is even or odd.

The conjugacy classes for D_{2n+1} are $\{e\}, \{r^{\pm 1}\}, \dots, \{r^{\pm(n-1)/2}\}$, and $\{sr^i : 0 \leq i < n\}$. The set with just the identity is always a conjugacy class, there are $(n-1)/2$ conjugacy classes that have 2 elements, a rotation and its inverse, and one class with all the reflections. For D_{2n} we have the almost the same conjugacy classes but the reflections are split into vertex reflections and midpoint reflections, as with an even number of vertices there are two types of reflections (lines of symmetry), vertex to vertex or midpoint to midpoint, as opposed to odd vertices where every line of symmetry is vertex to midpoint. The conjugacy classes of D_{2n} are thus $\{e\}, \{r^{\pm 1}\}, \dots, \{r^{\pm(n/2-1)}\}, \{sr^{2i+1} : 0 \leq i < \frac{n}{2}\}$, and $\{sr^{2i} : 0 \leq i < \frac{n}{2}\}$.

Question 2. Find all finite groups with exactly 2 conjugacy classes. For a finite group G to have exactly 2 conjugacy classes, one must be $\{e\}$ so the other is $G \setminus \{e\}$. If we consider the action of conjugation on G and an element $g \in G$, we have $|G| = |C_G(g)||K_g|$ by orbit-stabilizer, because the conjugacy classes K_g are the orbits of g and $C_G(g)$ are the stabilizers of g under the conjugation action. From this we can deduce that the size of each conjugacy class must divide $|G|$, and since $|G \setminus \{e\}| = |G| - 1$ this implies $|G| = 2$ because the only number n where $n-1 \mid n$ is $n = 2$. Thus all finite groups with exactly 2 conjugacy classes are all isomorphic to $(\mathbb{Z}_2, +)$, (the only distinct group of size 2). The classes are $\{0\}$ and $\{1\}$:D.

Question 3. Let G be a finite group and let $H \leq G, H \neq G$. Prove that G is not the union of the conjugates of H (equivalently, prove that there exists $g \in G$ such that g is not conjugate to an element of H). Let $|H| = n$ and $[G : H] = k$, then because G is finite, $|G| = kn$. There are at most k distinct conjugate of H , gHg^{-1} for $g \in G \setminus H$, and every conjugate set has the identity, we can bound the union of all conjugate sets:

$$\left| \bigcup_{g \in G} gHg^{-1} \right| \leq n(k-1) + 1 = nk - n + 1 < nk = |G|$$

This is assuming $n > 1$. We have $n(k-1) + 1$ because the conjugation of the identity by any element in G is the identity, so we subtract it from the possible distinct elements gained by conjugation and add it at the end so it is not double counted. This resolves that for finite G with subgroup H , $G \neq \bigcup_{g \in G} gHg^{-1}$ because you can not obtain all the elements in G from conjugation.

Question 4. Find all normal subgroups of S_n for $n \geq 5$. Claim: $\{e\}, S_n$, and A_n are the only normal subgroups of S_n . Proof: (similar to proof A_n is simple for $n \geq 5$) Assume there is another normal subgroup H . Then because normalcy is preserved under intersection, $H \cap A_n \subset A_n$ is also a normal subgroup, by simplicity of A_n , $H \cap A_n = \{e\}$ or A_n . If $H \cap A_n = A_n$, then $H = A_n$, otherwise H contains odd permutations, and then $H = S_n$, both contradict assumption. If $H \cap A_n = \{e\}$ then $H = \{e\}$ or H contains at least one odd permutation. Consider first at least two odd permutations $\sigma, \tau \in H$, then either $\sigma^2 \neq e$ or $\sigma\tau \neq e$ but both of these are even, contradicting assumption that $H \cap A_n = \{e\}$. Now consider $H = \{e, \sigma\}$ where σ is an odd permutation, but this is not a normal subgroup, as any normal subgroup with two elements must be in the center, but for $n \geq 3$, $Z(S_n) = \{e\}$ so we have reached the final contradiction and thus the only non-trivial normal subgroup of S_n is A_n .