

PS3

Question 1. Consider the polynomial ring $R = \mathbb{R}[x_1, x_2, x_3, x_4]$ in 4 variables. Define an action of S_4 on R by $\sigma(p(x_1, x_2, x_3, x_4)) = p(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)})$. Find the stabilizers of the following polynomials:

a) $x_1 + x_2$

b) $x_1x_2 + x_3x_4$

c) $x_1x_2x_3$

a) $x_1 + x_2$

For $p = x_1 + x_2$, the stabilizers of p are $\{(12), (12)(34), (34), ()\}$.

b) $x_1x_2 + x_3x_4$

For $p = x_1x_2 + x_3x_4$, the stabilizers of p are $\{(12)(34), (12), (34), (13)(24), (14)(23), ()\}$

c) $x_1x_2x_3$.

For $p = x_1x_2x_3$ the stabilizers of p are all of S_3 .

Question 2. For each of the following claims, either give a proof or find a counterexample:

a) The relation "being a subgroup of" is transitive; in other words, if $H \leq G$ and $K \leq H$, then $K \leq G$.

b) The relation "being a normal subgroup of" is transitive; in other words, if $H \triangleleft G$ and $K \triangleleft H$, then $K \triangleleft G$.

a) If $H \leq G$ and $K \leq H$ then $K \subseteq G$.

$$K \leq H \implies \forall k_1, k_2 \in K, k_1k_2 \in K \text{ and } \forall k \in K, k^{-1} \in K$$

Thus $K \leq G$

b) Counterexample: If we take $G = D_4$, $H = \langle rs, r^2 \rangle$, and $K = \langle rs \rangle$. $\langle rs, r^2 \rangle$ is a normal subgroup of D_4 (as seen in class), and since $|H|/|K| = [H : K] = 4/2 = 2$ so $K \triangleleft H$. But we also saw in class that $\langle rs \rangle$ is not a normal subgroup of D_4 .

Question 3. Use the first isomorphism theorem to construct an isomorphism between the quotient group $\mathbb{R}/\mathbb{Z}$ and S^1 (the norm 1 complex numbers as a group under multiplication).

If we construct a group homomorphism between $\mathbb{R}$ and $\mathbb{C}$ where $\mathbb{Z}$ is the kernel and S^1 the image, we may then use the first isomorphism theorem to prove and construct an isomorphism. Let $\varphi : \mathbb{R} \rightarrow \mathbb{C}$ defined by $\varphi(x) = e^{2\pi ix}$. This is a homomorphism because

$$\forall x_1, x_2 \in \mathbb{R}, \varphi(x_1 + x_2) = e^{2\pi i(x_1 + x_2)} = e^{2\pi ix_1} e^{2\pi ix_2} = \varphi(x_1)\varphi(x_2)$$

Using Euler's formula, if $x \in \mathbb{Z}$ then $e^{2\pi ix} = 1$ because it represents $|x|$ full rotations (clockwise or counter-clockwise depending on the sign), and thus $\ker \varphi = \mathbb{Z}$. The image of φ is S^1 , because $\varphi(x) = e^{2\pi ix} = \cos(2\pi x) + i \sin(2\pi x)$ where $|e^{2\pi ix}| \sqrt{\cos^2(2\pi x) + \sin^2(2\pi x)} = 1$, so the norm of $\varphi(x)$ is always 1 and it is in the complex plane.

Then using the First Isomorphism Theorem, there exists an isomorphism $\mathbb{R}/\mathbb{Z} \cong S^1$ defined by $\bar{\varphi}(\bar{x}) = \varphi(x)$ where $\bar{x} \in \mathbb{R}/\mathbb{Z}$.

Question 4. Let $H \leq G$ a subgroup. Let

$$N_G(H) = \{g \in G : \forall h \in H, ghg^{-1} \in H \wedge g^{-1}hg \in H\}$$

- a) Prove that $N_G(H)$ is a subgroup of G .
- b) Prove that $N_G(H)$ contains H .
- c) Prove that H is a normal subgroup of $N_G(H)$
- d) Find an example of G, H such that $N_G(H) = H$
- e) Suppose that $G = GL_2(\mathbb{R})$ and H is the subgroup of invertible diagonal matrices (so $H = \mathbb{R}^\times \times \mathbb{R}^\times$ as a group). Find $N_G(H)$ and describe the group $N_G(H)/H$.

a) Closure: $\forall g_1, g_2 \in N_G(H), \forall h \in H$

$$(g_1g_2)h(g_1g_2)^{-1} = (g_1g_2)h(g_2^{-1}g_1^{-1}) = g_1(g_2hg_2^{-1})g_1^{-1} \in H$$

$$(g_1g_2)^{-1}h(g_1g_2) = g_2^{-1}(g_1^{-1}hg_1)g_2 \in H$$

Then $g_1g_2 \in N_G(H)$.

Inverse: $\forall g \in N_G(H), \forall h \in H$

$$g^{-1}hg \in H \wedge ghg^{-1} \in H \implies g^{-1} \in N_G(H)$$

Thus $N_G(H) \leq G$.

b) Let $h_1 \in H$. $\forall h_2 \in H$, since $H \leq G$,

$$h_1h_2h_1^{-1}, h_1^{-1}h_2h_1 \in H \implies h_1 \in N_G(H)$$

Then $H \subseteq N_G(H)$.

c) $\forall h \in H, g \in N_G(H)$

$$ghg^{-1} \in H \text{ because } g \in N_G(H)$$

thus $H \triangleleft N_G(H)$.

d) $G = D_4, H = \langle s \rangle = \{s, e\}$ or any of the subgroups generated by a reflection, then $N_G(H) = H$ because conjugation by any other action will never give s (or e).

e) $H = \left\{ \begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix} : r_1, r_2 \in \mathbb{R}^\times \right\}$ and $G = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{R}, ad \neq bc \right\}$

Then for $g \in G : g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, g^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ so

$$\begin{aligned} ghg^{-1} &= \frac{1}{ad-bc} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\ &= \frac{1}{ad-bc} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} dr_1 & -br_2 \\ -cr_2 & ar_2 \end{bmatrix} = \frac{1}{ad-bc} \begin{bmatrix} (adr_1 - bcr_2) & (-abr_1 + abr_2) \\ (cdr_1 - cdr_2) & (-cbr_1 + dar_2) \end{bmatrix} \end{aligned}$$

For this to be in H we would need $cdr_1 = cdr_2$ and $abr_1 = abr_2$, since these must be conditions on g , we must have $b = c = 0$ or $a = d = 0$ (because $ad \neq bc$). So

$$N_G(H) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{R}, b = c = 0 \vee a = d = 0 \right\} = H \sqcup G \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

The quotient group $N_G(H)/H$ then has two elements and thus it is isomorphic to $\mathbb{Z}/2$.