

PS2 - Orbit-Stabilizer Theorem

Question 1. Define an action of S_n on $X := \{1, \dots, n\} \times \{1, \dots, n\}$ by $g \cdot (a, b) = (g(a), g(b))$. Find the orbits of this action and find the stabilizer of any point $(a, b) \in X$.

The orbits of this action are (a, b) s.t. $a = b$ and (a, b) s.t. $a \neq b$. This is because the symmetric group S_n acts on a tuple (a, a) as $(g(a), g(a))$ and we must have $g(a) = g(a)$ so this group action must send (a, a) to (b, b) for some $a, b \in \{1, \dots, n\}$, and s_n is transitive so you can send any element of $\{1, \dots, n\}$ to any other elements.

For $a \neq b$, since there cannot be a permutation in S_n that sends a and b to the same element because the permutations represent an injective map, also by transitivity you can send any element to any other element so this orbit covers all $(a, b) : a \neq b$.

The stabilizer of any point (a, b) contains all the permutations that do not permute a nor b .

Question 2. Use orbit-stabilizer theorem to prove that the number of k -element subsets of $\{1, \dots, n\}$ is $\frac{n!}{k!(n-k)!}$.

If we consider the set X of all the k -element subsets of $\{1, \dots, n\}$ and the group action $\varphi : S_n \times X \rightarrow X$ where for any $\sigma(\{x_1, \dots, x_k\}) = \{\sigma(x_1), \dots, \sigma(x_k)\}$ for any $x = \{x_1, \dots, x_n\}$. Then the number of k -element subsets of $\{1, \dots, n\}$ is the number of orbits of the action because each orbit will be permuting all the same elements, so it may represent a unique set of k elements.

The stabilizer of each of these elements $x \in X$ would be all the permutations that permute $n-k$ elements, which is $(n-k)!$ permutations, and since we don't care about the order of the elements in x , there are also $k!$ ways to permute this subset, giving us $k!(n-k)!$ as desired.

Then num. of orbits = $\frac{|S_n|}{|Stab_{(S_n)}(x)|} = \frac{n!}{k!(n-k)!}$ by orbit-stabilizer theorem.

Question 3. Consider the action of $\mathbb{R}^\times$ on $\mathbb{R}^n$ by $t \cdot (a_1, \dots, a_n) = (ta_1, \dots, ta_n)$. Find the orbits of this action.

The orbits of this action are each unique line going through the origin in the space, because for each vector the group action can only scale (or move) it to any other vector on the line.

Question 4. Let k be a finite field with q elements. By counting orbits for a group action, find the number of 1-dimensional subspaces of k^n .

If we consider $k^\times$ as a multiplicative group and then the action of $k^\times$ on $(k^\times)^n$, similar to the action in question 3, then the number of orbits of this action will be the number of 1-dimensional subspaces (because we are discounting the orbit with just the zero vector). To count these orbits we can use a corollary shown in class: For a group G and a set X where G acts on X , if X is finite then $\frac{|X|}{|G|} = |X/G|$ where $|X/G|$ is the number of orbits.

Then the number of orbits is $\frac{|(k^\times)^n|}{|k^\times|} = \frac{q^n - 1}{q - 1}$.

Question 5. Prove that the group of rotations of $\mathbb{R}^3$ that preserve a regular cube is isomorphic to S_4 .

If we consider the four diagonals of the cube connecting each opposite corner and number them 1-4, we may understand the rotations preserving the structure of the cube. We can use orbit-stabilizer theorem to show that there are 24 such rotations where the set is the 4 numbered diagonals and the group action is the rotations:

Since you can send each diagonal to any other while preserving structure and the action is transitive, the orbit is of size 4, and the stabilizer of any diagonal (or equivalently any pair of opposite corners) is just all the ways you can permute the other 3 opposite edges, which is $3! = 6$ times, this shows that there are 24 rotations of $\mathbb{R}^3$ that preserve the cube structure.

To show an isomorphism we may use an injective map (that then must be bijective as the sets are the same size) that sends the permutations $\sigma \in S_4$ to rotations that switch the diagonals that are permuted in σ . The inverse of this map sends the rotations of the cube to the permutation σ which matches the way the numbered diagonals are switched.

Question 6. Prove that there is no transitive action of S_3 on a set of 5 elements. Let X be our set of 5 elements. An action being transitive is equivalent to there being only one orbit, X . We can use orbit-stabilizer theorem to show that if some action was in fact transitive, we would require:

$$6 = |S_3| = |X||\text{Stab}_G(X)| = 5|\text{Stab}_G(X)|$$

But since $\text{Stab}_G(X)$ must be an integer, there is no way we can have this equality, because 5 does not divide 6. Thus for any action of S_3 on a set of 5 elements, there is always more than one orbit and equivalently, the action is not transitive.