

PS9

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1. (a) Find the character table of S_4 . [First find all the conjugacy classes in S_4 and use them to label the columns. Then start with the representations that you know.]
- (b) Using the character table decompose $V \otimes V$ into irreps where $V = \{(a_1, a_2, a_3, a_4) : a_1 + \dots + a_4 = 0\}$ is the standard representation.

(a)

	()	(12)(34)	(12)	(1234)	(123)
(Trivial rep.) T	1	1	1	1	1
(Sign rep.) σ	1	1	-1	-1	1
(Standard rep.) S	3	-1	1	-1	0
$S \otimes \sigma$	3	-1	-1	1	0
$\mathbb{C}^4$ perm. rep.	4	0	2	0	1
(Regular rep.) R	24	0	0	0	0

Note that $(\mathbb{C}^4, \text{perm}) = S \oplus T$, so it is reducible. Since the regular representation can be decomposed $\mathbb{C}^{S_4} = \bigoplus_i V_i^{\dim V_i}$ where $V_1, \dots, V_n$ is a complete list of irreducible representations of S_4 , there must be another representation W of degree 2 because $\dim T + \dim \sigma + \dim S^3 + \dim(S \otimes \sigma)^3 = 1 + 1 + 3^3 + 3^3 = 20$. So filling in the other columns:

	()	(12)(34)	(12)	(1234)	(123)
W	2	2	0	0	-1

And this representation W is irreducible because $\langle \chi_W, \chi_W \rangle = 1$.

(b) $V \otimes V = V \oplus \sigma \oplus T \oplus (S \otimes \sigma) \oplus W$.

2. Let G be a finite group and X a finite set with an action of G . Assume there exists $g \in G$ and $x \in X$ such that $g \cdot x \neq x$. Let $\mathbb{C}^X$ be the space of functions on X which has an action of G as before.

- (a) Suppose that we decompose $\mathbb{C}^\times = V_1^{\oplus m_1} \oplus \cdots \oplus V_n^{\oplus m_n}$ into irreducible representations. For any $g \in G$, let $\text{Fix}(g) = \{x \in X : g \cdot x = x\}$. Prove that

$$\sum_{g \in G} |\text{Fix}(g)|^2 = |G|(m_1^2 + \cdots + m_n^2)$$

- (b) Let $V = \{f \in \mathbb{C}^\times : \sum_{x \in X} f(x) = 0\}$. Prove that V is irreducible if and only if $\sum_{g \in G} |\text{Fix}(g)|^2 = 2|G|$.

- (a) We know that $\langle \chi_{\mathbb{C}^\times}, \chi_{\mathbb{C}^\times} \rangle = \sum_{i=1}^n m_i^2$, and that $\chi_{\mathbb{C}^\times}(g) = |\text{Fix}(g)|$. And $\langle \chi_{\mathbb{C}^\times}, \chi_{\mathbb{C}^\times} \rangle = \frac{1}{|G|} \sum_{g \in G} \overline{\chi_{\mathbb{C}^\times}(g)} \chi_{\mathbb{C}^\times}(g) = \frac{1}{|G|} \sum_{g \in G} |\text{Fix}(g)|^2$ so it follows that $\sum_{g \in G} |\text{Fix}(g)|^2 = |G|(m_1^2 + \cdots + m_n^2)$.

- (b) Suppose $\sum_{g \in G} |\text{Fix}(g)|^2 = 2|G|$, then $\mathbb{C}^\times = V \oplus W$ for some subrepresentation $W \subseteq \mathbb{C}^\times$, and then $\dim V = 1$ so V is irreducible. Now suppose that V is irreducible, then V is one of the irreducible representations in the decomposition of $\mathbb{C}^\times$.

3. Let p be a prime. Let $\mathbb{F}_p^\times$ act on $\mathbb{F}_p$ by multiplication and let $G = \mathbb{F}_p \rtimes \mathbb{F}_p^\times$, a group of size $p(p-1)$. Let $X = \mathbb{F}_p$. Define an action of G on X by $(b, a) \cdot x = ax + b$.

- (a) Let $V = \{f \in \mathbb{C}^\times : \sum_{x \in X} f(x) = 0\}$. Prove that V is irreducible.
(b) Find all 1-dimensional representations of G .
(c) Find all irreducible representations of G .

- (a) Consider $\text{Fix}(g)$ for three cases of $g = (b, a)$. First suppose $g = (0, 1)$ is the identity on G , then $|\text{Fix}(g)| = |\mathbb{F}_p| = p$. Now suppose $g = (b, 1)$, $b \neq 0$, then g fixes no element of X , so $|\text{Fix}(g)| = 0$. Now suppose $g = (b, a)$, $a \neq 1$, then g fixes only one $x \in X$ where $x = \frac{-b}{a-1}$, and since there are p options for b and $p-2$ options for a ($a \neq 1$), so for this case there are $p(p-2)$ options in total, then $\sum_{g \in G} |\text{Fix}(g)|^2 = |\text{Fix}(e)|^2 + \sum_{(b,1) \in G, b \neq 0} |\text{Fix}(g)|^2 + \sum_{(b,a) \in G, a \neq 1} |\text{Fix}(g)|^2 = p^2 + \sum_{(b,1) \in G, b \neq 0} 0 + \sum_{(b,a) \in G, a \neq 1} 1 = p^2 + p(p-2) = 2p^2 - 2p = 2(p(p-1)) = 2|G|$, so by 2(b) V is irreducible.

- (b) For $(a, b), (c, d) \in G$, $(a, b) \cdot (c, d) = (a + bc, bd)$, so for a one dimensional representation of G over $\mathbb{C}$, we have a homomorphism $\rho : G \rightarrow \mathbb{C}^\times$. If we consider homomorphisms that function as projectors onto $\mathbb{F}_p^\times$, then we have $\rho_k((b, 1)) = \zeta_{p-1}^k$ for $k = 1, \dots, p-1$, so there are $p-1$ unique 1-dimensional representation.

- (c) p conjugacy classes of size $p-1$ in G , so then there are p irreducible representations of G that are shown above.

4. Let G be a finite group and let V be a representation of G over $\mathbb{C}$. Suppose that $\chi_V(g) = 0$ for all $g \neq e$. Prove that $|G|$ divides $\dim V$.

We know that $\chi_V(e) = \dim V$, and from class $\frac{1}{|G|} \sum_{g \in G} \chi_V(g) = \dim V^G$ and since for $g \neq e$, $\chi_V(g) = 0$, so $\frac{1}{|G|} \sum_{g \in G} \chi_V(g) = \frac{\chi_V(e)}{|G|} = \frac{\dim V}{|G|} = \dim V^G$ and since $\dim V^G$ must be a natural number, $|G|$ must divide $\dim V$.

5. Consider the dihedral group D_n with n odd.

- (a) Find all 1-dimensional representations of D_n .
 - (b) Prove that all other irreducible representations of D_n are 2-dimensional. [Hint: you can use that D_n has $(n+3)/2$ conjugacy classes.]
- (a) The relations of D_n are $r^n = 1, s^2 = 1$, and $srs = r^{-1}$, the characters of the representations of D_n are defined by what $\chi(r)$ and $\chi(s)$ are. Let $\chi(r) = a$ and $\chi(s) = b$, then $a^n = 1 = b^2$ so $b = \pm 1$ (because we are considering only 1-dimensional representations) and $bab = a^{-1}$ implies that $a = a^{-1}$ thus $a = 1$ (because n odd), so there are two one dimensional representations of D_n
- (b) D_n has $(n+3)/2$ conjugacy classes and thus $(n+3)/2$ irreducible representations, so $(n-1)/2$ irreps left. Since there are $(n-1)/2$ conjugacy classes of $\langle r \rangle$, there are associated irreducible representations $\rho_k : D_n \rightarrow GL(V)$ where these correspond to the rotations matrix $\begin{bmatrix} \cos(2\pi k/n) & \sin(2\pi k/n) \\ \sin(2\pi k/n) & \cos(2\pi k/n) \end{bmatrix}$ So the rest of the irreducible representations of D_n are 2-dimensional.

Thank you for grading me :D