

PS8

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1. Let K be a field of characteristic p .

- (a) Let V be a K -vector space and let $T : V \rightarrow V$ be a linear operator such that $T^p = I$. Prove that there exists $v \in V, v \neq 0$ such that $T(v) = v$. [Hint: consider $(T - I)^p$.]
- (b) Let $G = \mathbb{Z}/p$. Prove that the only irreducible representation of G over K is the trivial representation.
- (c) Let G be a p -group. Prove that the only irreducible representation of G over K is the trivial representation. [Hint: use induction and the fact that every p -group contains a central copy of $\mathbb{Z}/p$.]

(a) $T^p = I$ and K is characteristic p , then $(T - I)^p = T^p + (-I)^p = I - I = 0$. So $0 = \det((T - I)^p) = (\det(T - I))^p$ thus $\det(T - I) = 0$ so 1 is an eigenvalue and there exists corresponding eigenvector $v \in V, v \neq 0$ such that $T(v) = 1 \cdot v = v$.

(b) Let V be a K vector space such that V is a non-trivial representation over G , and for any g , let $\rho(g) = T$ linear map $T : V \rightarrow V$ where $T^p = I$ because $(\rho(g))^p = \rho(g^p) = \rho(e) = I$. Then from (a) there exists a non-zero vector $v \in V$ such that $T(v) = v$, then there exists subrepresentation $W \subset V$ where $W = \text{span}(v)$. W is a subrepresentation because it is a subspace (it is the span of an eigenvector) and for any $g \in G, w \in W, g \cdot w = \rho(g) \cdot w = T(w) = \lambda w \in W$. Thus the only irreducible representation of G over K is the trivial representation, because in the trivial rep, $\rho(g) = I$ for all g .

(IDK if this above really works because this is supposing that for all $g, h \in G \rho(g) = \lambda \rho(h)$ for some $\lambda \in K$, but im not confident this is true)

This is an alternate proof:

Consider set $W = \{v \in V | \rho(1)v = v\}$, then this is closed under group action by G because $\forall g \in G, \forall v \in V \rho(1)\rho(g)v = \rho(g)\rho(1)v = \rho(g)v$. From (a) this subrep exists because $[\rho(1)]^p = I$ implies that there exists a non-zero vector $v \in V$ where $\rho(1)v = v$.

(c) Let $|G| = p^n$ for some $n \in \mathbb{N}$. Lets proceed by induction on n :

Base case: $n = 1$ proved in (b)

Inductive step: Suppose for $G_n, |G_n| = p^n$ the only irreducible representation over K is the trivial representation. Then consider $G_{n+1}, |G_{n+1}| = n+1$. Take $H = Z(G_{n+1})$, then $|H| \geq p$, want to show that H acts trivially on V , but since H is a p group smaller than G , then by the inductive hypothesis, the only irreducible representation of H over K is

the trivially representation, so then the map $G \rightarrow GL(V)$ is equivalent to $G \rightarrow G/H \rightarrow GL(V)$, and since G/H is a smaller p -group, the only irreducible representation of G/H over K is the trivial representation, indicating that the only irreducible representation of G over K is the trivial map by considering map above $G \rightarrow G/H \rightarrow GL(V)$.

2. Let V, W be two vector spaces, with $\dim V, \dim W \geq 2$. Prove that the map $V \times W \rightarrow V \otimes W$ given by $(v, w) \mapsto v \otimes w$ is neither injective or surjective.

Consider basis $v_1, \dots, v_n$ and $w_1, \dots, w_m$ of V and W respectively ($n, m \geq 2$). Then since for all $(v, 0) \in V \times W, (v, 0) \mapsto v \otimes 0 = 0$, so the map has a non-trivial kernel and is thus non-injective. Also since we have that $\dim(V \otimes W) = \dim(V) \cdot \dim(W)$, and $\dim(V \times W) = \dim(V) + \dim(W)$, if $\dim(V), \dim(W) > 2$, then the map can't be surjective simply because $\dim(V \times W) < \dim(V \otimes W)$, and if $\dim(V) = \dim(W) = 2$, then since the map is not injective (above), it can't be surjective (because the dimension of the image is less than the dimension of $V \otimes W$).

3. Let V be an irreducible representation of a group G and let W be a 1-dimensional representation. Prove that $V \otimes W$ is irreducible.

Let $v_1, \dots, v_n$ be a basis for V and let w be a basis for W , then $v_1 \otimes w, \dots, v_n \otimes w$ is a basis for $V \otimes W$. Suppose there is a non-trivial subrepresentation $X \subsetneq V \otimes W$, then the basis for X is a subset $\{v_i \otimes w | i \in I\} \subset \{v_1 \otimes w, \dots, v_n \otimes w\}$ with corresponding index I . Let $x = \sum_{i \in I} \lambda_i (v_i \otimes w) \in X$ then for all $g \in G, g \cdot x = \sum_{i \in I} \lambda_i (g \cdot v_i) \otimes (g \cdot w) \in X$ but this implies that V has a subrepresentation with basis $\{v_i | i \in I\}$, so this is a contradiction and $V \otimes W$ is irreducible. We can also see that since W is one dimensional $\text{Hom}_G(V^*, W) = V$, thus $V \otimes W \cong V$ and thus $V \otimes W$ is irreducible.

4. Let G be a group and let K be a field. Let $\rho : G \rightarrow GL(V)$ be an irreducible representation over K . Let $g \in G$.

- (a) Assume that $K = \mathbb{C}$ and that G is finite. Prove that $\rho(g)$ is diagonalizable.
- (b) Give an example of G, K, ρ, V, g as above with $K = \mathbb{C}, G$ not finite, and $\rho(g)$ not diagonalizable.
- (c) Give an example of G, K, ρ, V, g as above $K \neq \mathbb{C}, G$ finite, and $\rho(g)$ not diagonalizable.

- (a) $\rho(g)$ diagonalizable implies that there exists invertible matrix P and diagonal matrix D such that $\rho(g) = PDP^{-1}$. Since G is finite then every representation of G is unitary and thus diagonalizable.

- (b) $G = \mathbb{Z}, K = \mathbb{C}$ and let $\rho(g) = \begin{bmatrix} 1 & g \\ 0 & 1 \end{bmatrix}$ so for $g = 1, \rho(g) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and this is not diagonalizable.

- (c) $G = \mathbb{Z}/n, n \geq 3, K = \mathbb{R}, V = \mathbb{R}^2$ and $\rho([k]) = \begin{bmatrix} \cos(2\pi k/n) & \sin(2\pi k/n) \\ -\sin(2\pi k/n) & \cos(2\pi k/n) \end{bmatrix}$ rotation matrices, since ρ has no eigenvectors, the matrix is not diagonalizable for any $[k]$.

5. Let G be any group (not necessarily finite) and let V be a unitary representation of G over $\mathbb{C}$. Prove that V is the direct sum of irreducible subrepresentations.

If V is irreducible we are done, so suppose V is reducible and that $W \subsetneq V$ is a non-trivial subspace, then since V is unitary, $W^\perp$ is also a subrepresentation and V can be decomposed as a representations: $V = W \oplus W^\perp$, then if $W, W^\perp$ are irreducible we are done. Suppose not, then $W, W^\perp$ have nontrivial subrepresentations W_1, W_2 such that $W, W^\perp$ can be decomposed as above. Eventually the subrepresentations must be irreducible, because the dimension of V is assumed to be finite, so V can be decomposed $V = V_1^{\oplus m_1} \oplus \dots \oplus V_n^{\oplus m_n}$ where V_i are irreducible.

6. Let V be a representation of a group G containing two subrepresentations $W_1, W_2 \subset V$ such that $V = W_1 \oplus W_2$. Assume that both W_1 and W_2 are irreducible and that $W_1 \not\cong W_2$. Prove that the only subrepresentations of V are $0, W_1, W_2, V$.

Suppose V has another subrepresentation U , then for all $u \in U, u = (w_1, w_2)$ such that $w_1 \in W_1, w_2 \in W_2$ and for all $g \in G, g \cdot u = g \cdot (w_1, w_2) = (g \cdot w_1, g \cdot w_2) \in U$, so then there exists subrepresentations $W'_1 \subset W_1, W'_2 \subset W_2$ such that $W'_1 = \{w_1 : \exists w_2 \in W_2, (w_1, w_2) \in U\}$ and $W'_2 = \{w_2 : \exists w_1 \in W_1, (w_1, w_2) \in U\}$, but W_1, W_2 are irreducible.