

PS7

Sasha Redden

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1. Let X be a finite set with an action of a group G . Let F be a field. Let F^X denote the set of all functions $f : X \rightarrow F$ regarded as a vector space under pointwise addition and scalar multiplication. Define a representation of G on F^X by $(g \cdot f)(x) = f(g^{-1} \cdot x)$.

- (a) Prove that F^X really is a representation of G .
- (b) Assume that F has characteristic 0. Suppose that G has n orbits when acting on X and assume that all of these orbits contain at least 2 elements. Prove that F^X has at least 4^n subrepresentations.

- (a) W.T.S.: $g \cdot (h \cdot f) = (gh) \cdot f$. For $g, h \in G$, $g \cdot (h \cdot f)(x) = h \cdot f(g^{-1} \cdot x) = f(h^{-1}g^{-1} \cdot x) = f((gh)^{-1} \cdot x) = (gh \cdot f)(x)$.

- (b) Let $X = \{a_1, \dots, a_m\}$, consider F^X as a vector space with vectors $(f(a_1), \dots, f(a_m))$, then let O_i be an orbit such that $X = O_1 \sqcup \dots \sqcup O_n$. Then for each O_i , there are (at least) 4 unique subrepresentations:

- i. 0-rep: Subrepresentation is subset of vectors $(f(a_1), \dots, f(a_m))$ for each $a_i \in O_i$, $f(a_i) = 0$.
- ii. a-rep: $(f(a_1), \dots, f(a_m))$ for each $a_i \in O_i$, $f(a_i) = a \in F$.
- iii. 0-sum-rep: $(f(a_1), \dots, f(a_m))$ such that $a_{i1} + \dots + a_{ik} = 0 \forall a_{ij} \in O_i$
- iv. F-rep: $(f(a_1), \dots, f(a_m))$.

These are subrepresentations because action by $g \in G$ only permutes the elements inside each O_i , so for example, in the a-rep for O_i , if $(f(a_1), \dots, f(a_m))$ is acted on by any $g \in G$, the entries of the vector corresponding to the elements of the orbit are all a , and they are permuted between each other by action of g such that all of the same entries are still a . These subrepresentations for each orbit can be interchanged and combined with each other to obtain at least 4^n distinct subrepresentations.

2. Consider the action of $\mathbb{Z}/n$ on $\mathbb{C}^2$ via the usual rotation matrices, so

$$\rho([k]) = \begin{bmatrix} \cos(2\pi k/n) & \sin(2\pi k/n) \\ -\sin(2\pi k/n) & \cos(2\pi k/n) \end{bmatrix}$$

Decompose $\mathbb{C}^2$ into two one dimensional representations.

The eigen vectors of this matrix are $v_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}$ and $v_2 = \begin{bmatrix} i \\ 1 \end{bmatrix}$ with eigen values $\lambda_1 = \cos(2\pi k/n) + i \sin(2\pi k/n)$ and $\lambda_2 = \cos(2\pi k/n) - i \sin(2\pi k/n)$. Then $W_1 = \text{span}(v_1)$, $W_2 = \text{span}(v_2)$ are each one dimensional subspaces and since for each $w_i \in W_i$ is a scalar multiple of an eigen vector, $\rho([k]) \cdot w_i = \alpha \rho([k]) \cdot v_i = (\alpha \lambda_i) v_i \in W_i$ for scalar α , so W_1, W_2 are one dimensional subrepresentations such that $\mathbb{C}^2 = W_1 \oplus W_2$.

3. For any group G , let $\widehat{G}$ denote the set of all group homomorphism $\rho : G \rightarrow \mathbb{C}^\times$.
 - (a) Prove that $\widehat{G}$ forms a group under pointwise multiplication.
 - (b) Suppose that $|\widehat{G}| = 2$ and let χ denote the non-trivial element of $\widehat{G}$. Prove that $\chi(g) = \pm 1$ for each $g \in G$.
 - (c) Describe $\widehat{\mathbb{Z}/n}$.
 - (d) Suppose that G is abelian and finite. Prove that $G \cong \widehat{\widehat{G}}$.
 - (e) Describe $\widehat{\mathbb{Z}}$.
- (a) Identity: Let $\rho_{id} : G \rightarrow \mathbb{C}^\times$ such that $\rho_{id}(g) = 1 \forall g \in G$. Then for any $\rho \in \widehat{G}$, $\rho_{id} \cdot \rho = \rho = \rho_{id} \cdot \rho$.
 Inverse: For any $\rho \in \widehat{G}$ Suppose $\rho(g) = a \in \mathbb{C}^\times$, then $\rho^{-1} \in \widehat{G}$ such that $\rho^{-1}(g) = 1/a$ is the inverse for ρ .
 Associativity: Follows from associativity of $\mathbb{C}^\times$.
- (b) Since $\widehat{G}$ is a group under pointwise multiplication and $|\widehat{G}| = 2$, $\chi^{-1} = \chi$. Then $\chi^2 = \chi \cdot \chi^{-1} = \rho_{id} = \chi^{-1} \cdot \chi$ so for $g \in G$, $\chi^2(g) = 1$ so $\chi(g) = \pm 1$.
- (c) Since $\mathbb{Z}/n$ is generated by 1, and $1 + \dots + 1 = 1^n = 0$, for every $\rho \in \widehat{G}$, $\rho(0) = \rho(1^n) = \rho^n(1) = 1$, so then for each $j \in \mathbb{N}$, $\rho_j(1) = e^{2\pi i \cdot j/n}$. ρ_j is a homomorphism because $\rho_j(k+m) = e^{2\pi i \cdot (k+m)/n} = e^{2\pi i \cdot k/n} \cdot e^{2\pi i \cdot m/n} = \rho_j(k) \cdot \rho_j(m)$. Also, $\rho_{j+n} = \rho_j$ so there are n unique homomorphisms ρ_j for $j \in \mathbb{Z}/n$, thus there is bijection $\varphi : \mathbb{Z}/n \rightarrow \widehat{\mathbb{Z}/n}$ where $\varphi(m) = \rho_m$. This is an isomorphism because for any $g \in \mathbb{Z}/n$, $\rho_{k+m}(g) = e^{2\pi i \cdot (k+m)g/n} = e^{2\pi i \cdot kg/n} \cdot e^{2\pi i \cdot mg/n} = \rho_k(g) \cdot \rho_m(g)$ for any $m, k \in \mathbb{Z}/n$, so $\varphi(m+k) = \rho_{m+k} = \rho_m \cdot \rho_k = \varphi(m) \cdot \varphi(k)$. Thus $\mathbb{Z}/n \cong \widehat{\mathbb{Z}/n}$.
- (d) Let G be abelian and finite, then $G = \mathbb{Z}/m_1 \times \dots \times \mathbb{Z}/m_n$ for $m_i \in \mathbb{N}$, from (c), $\mathbb{Z}/n \cong \widehat{\mathbb{Z}/n}$ so $G \cong \widehat{\mathbb{Z}/m_1} \times \dots \times \widehat{\mathbb{Z}/m_n}$. Want to show: $\widehat{\mathbb{Z}/m_1} \times \dots \times \widehat{\mathbb{Z}/m_n} \cong \mathbb{Z}/m_1 \times \dots \times \mathbb{Z}/m_n$. Consider function $\phi : \widehat{\mathbb{Z}/m_1} \times \dots \times \widehat{\mathbb{Z}/m_n} \rightarrow \mathbb{Z}/m_1 \times \dots \times \mathbb{Z}/m_n$ such that $\phi((\rho_1, \dots, \rho_n)) = \rho_{(1, \dots, n)}$ where for $g = (a_1, \dots, a_n) \in G$, $\rho_{(1, \dots, n)}(g) = (\rho_1(a_1), \dots, \rho_n(a_n))$. Note this function is clearly injective and surjective and ϕ is a homomorphism because for any $(\tau_1, \dots, \tau_n), (\rho_1, \dots, \rho_n) \in \widehat{\mathbb{Z}/m_1} \times \dots \times \widehat{\mathbb{Z}/m_n}$, $\phi((\tau_1, \dots, \tau_n) \cdot (\rho_1, \dots, \rho_n)) = \phi((\tau_1 \cdot \rho_1, \dots, \tau_n \cdot \rho_n)) = (\tau \cdot \rho)_{(1, \dots, n)} = \tau_{(1, \dots, n)} \cdot \rho_{(1, \dots, n)} = \phi((\tau_1, \dots, \tau_n)) \cdot \phi((\rho_1, \dots, \rho_n))$. Thus $G \cong \widehat{\widehat{G}}$.

- (e) We have seen that $\widehat{\mathbb{Z}}$ is the profinite integers, or the profinite completion of the integers, such that $\widehat{\mathbb{Z}} = \varprojlim \mathbb{Z}/n\mathbb{Z}$.